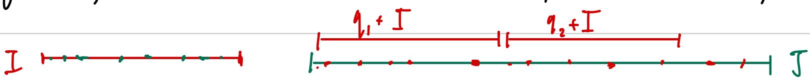
Measure Theory with Ergodic Horizons
Lecture 11

Examples of ergodic equivalence relations.

(a) The Vitali equivalence relation E_V on $(\mathbb{R}, \lambda)$. Recall that $x E_V y :\Leftrightarrow x-y \in \mathbb{Q}$, i.e. E_V is the orbit equivalence relation of the translation action $\mathbb{Q} \curvearrowright \mathbb{R}$. Let $\lambda :=$ Lebesgue measure.

Prop. E_V is λ-ergodic.

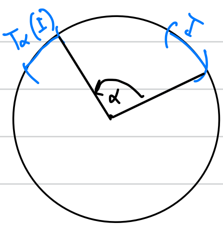
Proof. Suppose not, so there is a partition $\mathbb{R} = A \sqcup B$ into E_V-invariant sets of positive measure. Then by the 99% lemma, $\exists$ interval J whose 99% is B, i.e. $\lambda(B \cap J)/\lambda(J) \geq 0.99$. Again by the 99% lemma, $\exists$ interval I, with $|I| \leq |J|$, whose 99% is A.



By the density of $\mathbb{Q}$, $\exists$ translates $q_1+I, q_2+I, \dots, q_k+I$, where $k = \lfloor \frac{|J|}{|I|} \rfloor$, $q_i \in \mathbb{Q}$, that pairwise disjoint and $\bigsqcup_{i=1}^{k}(q_i+I) \subseteq J$ is at least half of J.

But q_i+I is 99% q_i+A and $q_i+A = A$ by E_V-invariance, so 99% of q_i+I is A. Thus, 99% of $\bigsqcup_{i=1}^{k}(q_i+I)$ is A but on the other hand, $100\% - 2 \cdot 1\% = 98\%$ of it is B, a contradiction. $\square$

(b) Irrational rotations. Identifying the unit circle $S^1 \subseteq \mathbb{R}^2$ with $\mathbb{R}/\mathbb{Z} \cong [0,1)$, we copy the Lebesgue measure from $[0,1)$ to S^1, and still denote it by λ. Then λ is rotation-invariant, i.e. for each angle $\alpha \in \mathbb{R}$, letting $T_\alpha : S^1 \to S^1$ by rotating every $x \in S^1$ by $2\pi\alpha$, we see that T_α preserves λ. We call T_α a rational/irrational rotation if α is rational/irrational.

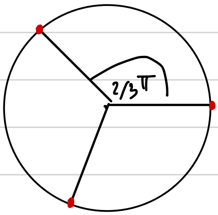


Prop. For each $\alpha \in \mathbb{R}$,

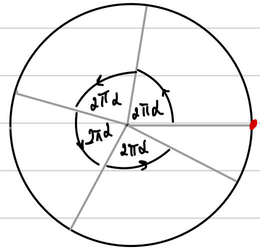
(i) α is irrational $\Leftrightarrow$ all orbits are dense in S^1 $\Leftrightarrow$ all orbits are infinite.

(ii) α is irrational $\iff$ T_α is λ-ergodic (i.e. its orbit eq. rel. is λ-ergodic).

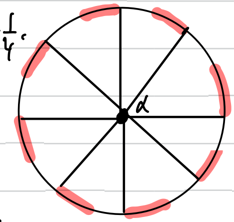
Proof. (i) Firstly, it's clear that if $\alpha = \frac{n}{m}$, where $\frac{n}{m}$ is reduced, then each T_α orbit has $\le m$ elements.



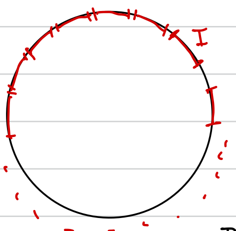
If α is irrational then each orbit is dense, which follows using the Euclidean algorithm, and is left as an exercise.



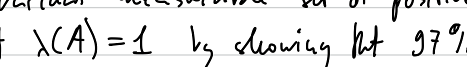
(ii) $\Leftarrow$. We show the contrapositive. Let α be rational, e.g. $\alpha = \frac{1}{4}$. Then letting $A := \bigcup_{n \in \mathbb{Z}} T_\alpha^n([0, \alpha/2))$ is T_α-invariant and has measure $1/2$.



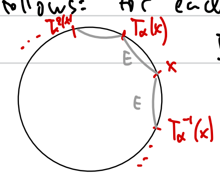
$\Rightarrow$. Suppose α is irrational, hence each orbit is dense. Let $A \subseteq S^1$ be an T_α-invariant measurable set of positive measure. We will show that $\lambda(A) = 1$ by showing that 97% of S^1 is A. By the 99% lemma, there is an interval-segment I whose 99% is A, and moreover, $\lambda(I) \le 0.01$ (i.e. 1% of S^1). By the density of the orbit of one of the endpoints of I, we can cover 98% of S^1 by finitely-many pairwise disjoint translates $T_\alpha^{n_1}(I), T_\alpha^{n_2}(I), \dots, T_\alpha^{n_k}(I)$, using that I has length $\le 1\%$ of S^1. The 99% of each $T_\alpha^{n_i}(I)$ is $T_\alpha^{n_i}(A) = A$, so the 99% of $\bigsqcup_{i=1}^{k} T_\alpha^{n_i}(I)$ is A, hence $\lambda(A) \ge 0.98 \cdot 0.99 \ge 0.97$. $\square$



The action of T_α on S^1 can be presented as an action of $\mathbb{Z}$ on S^1 where $1 \in \mathbb{Z}$ acts as T_α, so $n \in \mathbb{Z}$ acts as T_α^n. Just like $\mathbb{Z}$ has its Cayley graph $\mathrm{Cay}(\mathbb{Z}) := (\mathbb{Z}, E)$, where $E = \{(x,y) : |x-y| = 1\}$.



Using this, we can define the Schreier graph G_α of this action $\mathbb{Z} \overset{T_\alpha}{\curvearrowright} S^1$ as follows: for each $x, y \in S^1$, put an edge $(x,y) \in E(G_\alpha) :\iff y = T_\alpha(x)$ or $x = T_\alpha(y)$.

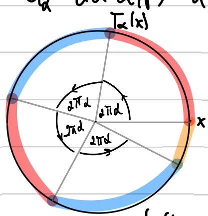


If α is irrational, then each connected component of G is exactly a T_α-

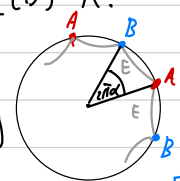
orbit and is isomorphic to $\mathrm{Cay}(\mathbb{Z})$, i.e. a bi-infinite line.
Since $\mathrm{Cay}(\mathbb{Z})$ can be properly coloured by 2 colours, we can also properly colour G by using Axiom of Choice and getting a transversal Y for the orbit eq. rel. E_{T_α} and colouring it red and then colouring $T_\alpha^{2k}(Y)$ red and $T_\alpha^{2k+1}(Y)$ green. But Y is non-measurable (as we will see below), so this colouring of S^1 is non-measurable.

Cor. For any irrational $\alpha \in \mathbb{R}$, G_α admits a measurable 3-colouring, but doesn't admit a measurable 2-colouring.

Proof.



This is a measurable 3-colouring. To see that there isn't a measurable 2-colouring, suppose there is: $S^1 = A \sqcup B$, where A, B are measurable and $T_\alpha(A) = B$ and $T_\alpha(B) = A$. These sets A, B are T_α^2-invariant, but $T_\alpha^2 = T_{2\alpha}$ and 2α is still irrational, hence $T_{2\alpha}$ is ergodic, so A, B are null or conull. But T_α is measure-preserving so $\lambda(A) = \lambda(T_\alpha(A)) = \lambda(B)$ hence $\lambda(A) = \lambda(B) = 1/2$, a contradiction.



$\square$

In particular, any transversal Y is non-measurable because otherwise it would give a measurable 2-colouring of G_α.

(c) Eventual equality $\mathbb{E}_0$ on $(2^{\mathbb{N}}, \mu_p)$ for all $p \in (0,1)$. Let $\mathbb{E}_0$ be the equivalence relation on $2^{\mathbb{N}}$ of eventual equality, i.e. $x \,\mathbb{E}_0\, y :\iff \forall^\infty_n\, x(n) = y(n) \iff \exists m \,\forall n \geq m\; x(n) = y(n)$.
Let μ_p be the Bernoulli(p) measure on $2^{\mathbb{N}}$.

Prop. $\mathbb{E}_0$ is μ_p-ergodic.

Proof. Is left as a HW exercise. We just mention here that $\mathbb{E}_0$ is the orbit eq. rel. of the action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ on $2^{\mathbb{N}} \cong \prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. $\square$

The phenomenon that any transversal for $\mathbb{E}_0$ and for the irrational rotation is non-measurable is a general phenomenon due to ergodicity:

Prop. Let $(X, \mathcal{B}, \mu)$ be an atomless probability space and let $\Gamma \curvearrowright X$ be an action of a ctbl group Γ where each $\gamma \in \Gamma$ maps sets in $\mathcal{B}$ to sets in $\mathcal{B}$.
If this action is μ-ergodic, then any transversal for its orbit eq. rel. E_Γ is non-measurable.

Proof. HW.